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A phenomenological model for the passive scalar variance transformation in a turbulent fluid

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Abstract—The exact transport equation for the passive contaminant concentration variance in incompressible turbulent fluid, derived by Corrsin, *J. appl. Phys.* **23,** 113–118 (1952), is expressed in the integro-differential form. The phenomenological integro-differential model equation for the variance space–time transformation and the general form of its solution are proposed. On the grounds of a few simple special cases it is shown that the solutions of the model equation follow some previously derived general restrictions [*J. Fluid Mech.* **91,** 337–355 (1979); **94,** 83–101 (1979)]. Relying on the constructed model some earlier approaches [*J. appl. Phys.* **23,** 113–118 (1982); *Adv. Geophys.* **6,** 117–137 (1959)] to the problem can be reinterpreted. The model is in good agreement with the available data of field and laboratory experiments.

1. INTRODUCTION

THE CONCENTRATION C in a cloud of a diffusing contaminant in the turbulent environment is often many times higher or lower than the expected mean concentration $\langle C\rangle$. Therefore, the description of the contaminant concentration field transformation at the mean concentration level only may prove insufficient. In such cases the fluctuations of the concentration field $C' = C-\langle C\rangle$ should be taken into account. One of the simplest quantity reflecting the fluctuational variability of the concentration field is the variance of concentration fluctuations $W = \langle C'^2\rangle$. The exact transport equation for the variance of a passive contaminant concentration in incompressible fluid was derived by Corrsin [1]:

$$\frac{\mathrm{D}}{\mathrm{D}t}W(\mathbf{x},t) = E(\mathbf{x},t)+\nabla\cdot[k_{\mathrm{m}}\nabla W(\mathbf{x},t)+\mathbf{h}(\mathbf{x},t)]-\chi(\mathbf{x},t) \tag{1}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t}+\langle\mathbf{u}\rangle\cdot\nabla;$$

$\mathbf{u}$ is the fluid velocity; t the diffusion time; $\mathbf{x} = (x_1, x_2, x_3)$ the vector of coordinates; k_{m} the molecular diffusivity; $\chi = 2k_{\mathrm{m}}\langle\nabla C'\cdot\nabla C'\rangle$ the rate of variance dissipation; and E the rate of variance production

$$E = 2\mathbf{j}\cdot\nabla\langle C\rangle, \tag{2}$$

$\mathbf{j} = \langle\mathbf{u}'C'\rangle$ and $\mathbf{h} = \langle\mathbf{u}'C'^2\rangle$ are the diffusive fluxes of the contaminant and the variance, respectively, and the angular brackets $\langle\ \rangle$ denote averaging.

In phenomenological models of turbulent diffusion the contaminant diffusive flux $\mathbf{j}$ is usually parametrized by the gradient-type relation,

$$\mathbf{j} = -\mathscr{K}^{\mathrm{C}}\cdot\nabla\langle C\rangle, \tag{3}$$

where $\mathscr{K}^{\mathrm{C}}$ is a tensor of eddy diffusivities for the contaminant. The tensor $\mathscr{K}^{\mathrm{C}}$ generally depends on the diffusion time as well as on the coordinates. By analogy with (3) it seems natural to introduce the following gradient-type relation to the variance flux $\mathbf{h}$:

$$\mathbf{h} = -\mathscr{K}^{\mathrm{W}}\cdot\nabla W, \tag{4}$$

where $\mathscr{K}^{\mathrm{W}}$ is a tensor of eddy diffusivities for the variance. The tensors $\mathscr{K}^{\mathrm{C}}$ and $\mathscr{K}^{\mathrm{W}}$ are introduced independently of each other and need not be equal. The exact equality of $\mathscr{K}^{\mathrm{C}}$ and $\mathscr{K}^{\mathrm{W}}$ has been proved by Meshcheryakov [5] only in the case when molecular diffusion effects are absent ($k_{\mathrm{m}} = 0$).

For the first time relations (3) and (4) were applied to the problem by Csanady [6, 7]. Assuming that the variance dissipation rate χ in equation (1) is proportional to the variance itself,

$$\chi = FW, \tag{5}$$

where F is the proportionality coefficient, and that the mean concentration field $\langle C\rangle$ as well as $\mathscr{K}^{\mathrm{C}} = \mathscr{K}^{\mathrm{W}}$ and F are known, Csanady got a closed equation for the concentration variance. Unfortunately the results of some experiments [8, 9] shows that Csanady's model is not widely applicable.

On the basis of the order of magnitude analysis for the contaminant conservation equation Chatwin and Sullivan [2, 3] have recently established three general features of the variance field evolution:

(i) the variance of an initially homogeneous concentration field increases with the diffusion time and, in case molecular diffusion effects are present ($k_{\mathrm{m}} \neq 0$), reaches a maximum value, thereafter asymptotically tending to zero;

(ii) the concentration fluctuation field depends considerably on the finite volume occupied by the contaminant at the initial diffusion time moment;

(iii) for sufficiently long diffusion times, if $k_{\mathrm{m}} = 0$, the

NOMENCLATURE

a, b	constants
C	contaminant concentration
E	production rate of the concentration variance
F	coefficient of the variance dissipation
f	arbitrary function
G	Green function
$\mathbf{h}$	diffusive flux of the variance
$\mathcal{I}$	unit tensor
I_0	Bessel function
$\mathbf{j}$	diffusive flux of the contaminant
$\mathcal{K}^{\mathrm{C}}, \mathcal{K}^{\mathrm{W}}$	tensors of the contaminant and the variance turbulent diffusivities
$K^{\mathrm{C}}, K^{\mathrm{W}}$	scalar diffusivities for the contaminant and the variance
k_{m}	molecular diffusivity
M, M_1, M_2	constants, defined by equations (16), (31) and (37)
m, n	constants
P, P_1, P_2	defined by equations (31)–(33)
p	emission rate of the variance source
q	emission rate of the contaminant source
r, r_0	radial coordinates of cyclindrical coordinate system
S	defined by equation (45)
t	time
$U, \mathbf{u}$	Eulerian velocity of the fluid
V	whole volume occupied by the fluid
W	variance of the contaminant concentration
$\mathbf{x}$	vector of coordinates, (x_1, x_2, x_3)
x, x''	distance downstream from the source.

Greek symbols

α, β, γ	constants
Γ	defined by equation (46)
δ	Dirac delta function
ζ	dimensionless characteristic width of the contaminant plume
θ	dissipation function
κ	constant
$\boldsymbol{\xi}$	vector of coordinates, (ξ_1, ξ_2, ξ_3)
ρ	dimensionless distance from the plume axis
$\sigma_{\mathrm{C}}^2, \sigma_{\mathrm{W}}^2$	spatial distribution variances of the contaminant and the concentration variance
σ_0	effective linear scale of the contaminant source
τ	time
ϕ	defined by equation (30)
φ	angular coordinate
χ	dissipation rate of the concentration variance
Ψ	initial spatial distribution of the variance
ψ	defined by equation (43).

Superscripts

$*$	denotes the space–time density
$'$	denotes the fluctuational component.

Subscripts

1, 2, 3	denote the components of the vector quantity.

distribution of variance is determined mainly by the turbulent diffusion process bearing a negligibly small influence from the variance production process.

These three important restrictions of the variance field transformation should naturally be reflected by any model intended to describe the variance field of the scalar concentration in a turbulent flow. Csanady's model does not reveal these restrictions.

For Chatwin and Sullivan the last of the three abovementioned effects served as a reason for postulating a specific 'core-bulk structure' of the variance field. The alternate resolution of the problem—essentially the whole variance production takes place in the initial interval of the diffusion time—was mentioned only marginally. Nevertheless, the latter alternative explanation seems to be more realistic.

In the following sections an attempt is made to construct a phenomenological model for the passive scalar concentration variance transformation in a turbulent fluid that follow the three restrictions given and agree with the available experimental data.

2. INTEGRO-DIFFERENTIAL MODEL

In view of the subsequent discussion it is important to point out that the variance production term E in equation (1) plays the role of the space–time distribution of variance sources productivities. Let us assume that the variances generated at separate space–time points retain to some extent their individualities which can be distinguished by space points and time moments of their generation. Let us denote the variance space–time density at a point $\mathbf{x}$ and time moment t, produced by a variance source at a space point $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ at a moment τ, by $W^*(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$. Then the field of the total variance W at a space–time point $(\mathbf{x}, t)$ is the integral of W^* over $\boldsymbol{\xi}$ and τ $(0 \leqslant \tau \leqslant t)$:

$$W(\mathbf{x}, t) = \int_0^t \int_V W^*(\mathbf{x}, t; \boldsymbol{\xi}, \tau)\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\tau, \tag{6}$$

where V is the volume of the whole fluid-occupied space and $\mathrm{d}\boldsymbol{\xi} = \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$. The total flux of variance $\mathbf{h}$ and the total dissipation rate χ can be expressed in a manner similar to (6) as integrals of the variance flux density $\mathbf{h}^*$

and the dissipation rate density χ^*, respectively,

$$\mathbf{h}(\mathbf{x},t) = \int_0^t \int_V \mathbf{h}^*(\mathbf{x},t;\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau, \qquad (7)$$
$$\chi(\mathbf{x},t) = \int_0^t \int_V \chi^*(\mathbf{x},t;\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

Substituting expansions (6) and (7) into the variance transport equation (1) we can obtain the integro-differential transport equation for W^*. It should be pointed out here that the representations (6) and (7) and consequently the integro-differential equation for W^* result as a special case from a general deductive (axiomatic) approach proceeding from the split-up of the averaging operator influence into the sequence of actions of a number of averaging operators [10–12]. The deductive approach confirms that the representation of the variance field W in the form (6) does not exclude the possibility of interactions between different variance-density fields W^* at a fixed space–time point $(\mathbf{x}, t)$, yet as the result of different interactions the value of total variance W has to be preserved.

Further, it seems natural to apply parametrizations similar to (4) and (5) to the densities $\mathbf{h}^*$ and χ^*:

$$\mathbf{h}^*(\mathbf{x},t;\xi,\tau) = \mathscr{K}^{\mathrm{W}}(\mathbf{x},t;\xi,\tau)\cdot\nabla W^*(\mathbf{x},t;\xi,\tau), \qquad (8)$$
$$\chi^*(\mathbf{x},t;\xi,\tau) = F(\mathbf{x},t;\xi,\tau)W^*(\mathbf{x},t;\xi,\tau).$$

In recent years considerable progress has been achieved in modelling the behaviour of the dissipation rate χ in decaying homogeneous turbulence. Attention has been payed mainly to the dissipation of homogeneous field of scalar fluctuations (for example [13]). The aim of the present paper is to show some peculiarities of the integral representations (6) and (7) particularly in substantially inhomogeneous scalar fluctuation fields. It justifies the assignment of one of the most simple rational closure approximation to the dissipation rate in (8).

The expressions for the total flux of the variance $\mathbf{h}$ and for the total dissipation rate χ now have the integral form:

$$\mathbf{h}(\mathbf{x},t) = \int_0^t \int_V \mathscr{K}^{\mathrm{W}}(\mathbf{x},t;\xi,\tau)\cdot\nabla W^*(\mathbf{x},t;\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau, \qquad (9)$$
$$\chi(\mathbf{x},t) = \int_0^t \int_V F(\mathbf{x},t;\xi,\tau)W^*(\mathbf{x},t;\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

The final closure assumptions (9) include the possibility that the variances generated in separate space–time points (ξ, τ) can pass through different transformations and may have different physical qualities in a given space–time point $(\mathbf{x}, t)$. These different qualities may lie, for instance, in differences of space–time scales of concentration fluctuations that form the density W^*. All possible differences in transformation of W^* that proceed from separate points (ξ, τ) are phenomenologically summed up into different diffusivities $\mathscr{K}^{\mathrm{W}}$ and dissipation coefficients F in our present closure assumptions (9). In a special case, if $\mathscr{K}^{\mathrm{W}}$ and F do not depend on ξ and τ, expressions (9) reduce to the ordinary non-integral closure assumptions (4) and (5).

Substituting expansion (6) and expressions (9) into the transport equation (1) we come to the integro-differential equation for W^*,

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_0^t \int_V W^*(\mathbf{x},t;\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$
$$= E(\mathbf{x},t) + \int_0^t \int_V \{\nabla\cdot[k_{\mathrm{m}}\nabla + \mathscr{K}^{\mathrm{W}}(\mathbf{x},t;\xi,\tau)\cdot\nabla]$$
$$- F(\mathbf{x},t;\xi,\tau)\}W^*(\mathbf{x},t;\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \qquad (10)$$

The general expression for the solution of the set (6) and (10) is

$$W(\mathbf{x},t) = \int_0^t \int_V [\Psi(\xi)\delta(\tau) + E(\xi,\tau)]$$
$$\times\, G(\mathbf{x},t;\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau, \qquad (11)$$

where $\Psi(\mathbf{x}) = W(\mathbf{x}, 0)$ is the spatial distribution of the variance at the initial moment $(t = 0)$ of the diffusion time, $\delta(t)$ is the Dirac delta function and G is the Green function, i.e. it satisfies the equation

$$\frac{\mathrm{D}}{\mathrm{D}t}G = \{\nabla\cdot[k_{\mathrm{m}}\nabla + \mathscr{K}^{\mathrm{W}}\cdot\nabla] - F\}G \qquad (12)$$

and the condition

$$\lim_{\tau\to t}\int_V f(\xi,\tau)G(\mathbf{x},t;\xi,\tau)\,\mathrm{d}\xi = f(\mathbf{x},t), \qquad (13)$$

where f is an arbitrary function of the coordinates and time.

3. UNIFORM GRADIENT OF THE MEAN CONCENTRATION

Corrsin [1] showed that in the one-dimensional steady and homogeneous turbulence, with $\langle\mathbf{u}\rangle = 0$, the initially linear distribution of the mean concentration

$$\langle C\rangle = mx \qquad (14)$$

where x is a coordinate, m a constant, is preserved for all time. From the statistical analysis of the scalar field with the mean concentration (14) Corrsin concluded that in the absence of the molecular diffusion $(k_{\mathrm{m}} = 0)$ the variance of concentration increases indefinitely with the diffusion time:

$$W = m^2\sigma_{\mathrm{C}}^2(t), \qquad (15)$$

where σ_{C}^2 has the sense of the mean square displacements of contaminant particles. On the other hand, assuming that the variance reaches some equilibrium value in the limit $t \to \infty$ it follows that $(k_{\mathrm{m}} \neq 0)$

$$W = Mm^2, \qquad (16)$$

where M is a constant. Attempts to verify experimentally the last two results (15) and (16) have been made by Wiskind [14], Alexopoulos and Keffer [15] and Sullivan [16].

It can be readily shown that our model is in agreement with the above-mentioned results (15) and

(16). In the present one-dimensional case the tensors $\mathcal{K}^{\mathrm{C}}$ and $\mathcal{K}^{\mathrm{W}}$ are reduced to the scalars K^{C} and K^{W}. Assuming that $k_{\mathrm{m}} \ll K^{\mathrm{W}}$ and that due to the turbulence homogeneity and steadiness $K^{\mathrm{C}} = K^{\mathrm{C}}(t)$, $K^{\mathrm{W}} = K^{\mathrm{W}}(t-\tau)$, $F = F(t-\tau)$, the solution of the one-dimensional analogue of equation (10) with assumption (14) has the form of

$$W = m^2 \int_0^t \frac{\mathrm{d}\sigma_{\mathrm{C}}^2(\tau)}{\mathrm{d}\tau} \frac{\theta(t-\tau)}{\theta(0)} \mathrm{d}\tau, \tag{17}$$

where $\sigma_{\mathrm{C}}(t)$ is determined in terms of $K^{\mathrm{C}}(t)$ by the relations

$$K^{\mathrm{C}}(t) = \frac{1}{2}\frac{\mathrm{d}\sigma_{\mathrm{C}}^2(t)}{\mathrm{d}t}, \quad \sigma_{\mathrm{C}}(0) = 0, \tag{18}$$

and

$$\theta(t) = \exp\left\{-\int F(t)\,\mathrm{d}t\right\}. \tag{19}$$

If the variance dissipation is absent, $\theta(t) = \theta(0)$, the result (15) follows from (17). Supposing that the variance reaches some equilibrium value in the limit $t \to \infty$ in the case $k_{\mathrm{m}} \neq 0$, the second Corrsin's result (16) follows from (17), but now the constant M has the value

$$M = \lim_{t\to\infty} \int_0^t \frac{\mathrm{d}\sigma_{\mathrm{C}}^2(\tau)}{\mathrm{d}\tau} \frac{\theta(t-\tau)}{\theta(0)} \mathrm{d}\tau, \tag{20}$$

if the limit on the right of (20) does exist.

Generally the asymptotic behaviour of the variance W depends on the nature of the variance dissipation, and may vary if the functions $\theta(t)$ and $\sigma_{\mathrm{C}}(t)$ with different asymptotic behaviour are applied in (17). Hence for the appropriate functions $\theta(t)$ and $\sigma_{\mathrm{C}}(t)$ the asymptote $W \sim t^{1/2}$, followed from a stochastic model by Durbin [17], can be reached as well.

4. A STEADY CONTAMINANT PLUME

Our discussion is now concerned with a special case of a steady contaminant plume in homogeneous turbulence in the half-space $x_3 > 0$. The contaminant source of the constant emission rate q is situated on the surface $x_3 = 0$ in the origin of Cartesian coordinates. The environment mean velocity U is directed along the coordinate axis x_1. The tensors $\mathcal{K}^{\mathrm{C}}$ and $\mathcal{K}^{\mathrm{W}}$ are isotropic, $\mathcal{K}^{\mathrm{C}} = K^{\mathrm{C}}\mathcal{I}$, $\mathcal{K}^{\mathrm{W}} = K^{\mathrm{W}}\mathcal{I}$ ($\mathcal{I}$ is unit tensor). Assume that K^{C} and K^{W}, F are the functions of x_1 and $(x_1-\xi_1)$, respectively, $K^{\mathrm{C}} = K^{\mathrm{C}}(x_1)$, $K^{\mathrm{W}} = K^{\mathrm{W}}(x_1-\xi_1)$, $F = F(x_1-\xi_1)$. The plume is seen to be 'slender', i.e. the diffusive fluxes of the contaminant and variance along the mean velocity direction are vanishingly small in comparison with the advection. Hence we formally accept $j_1 = h_1 = 0$. The molecular diffusivity k_{m} is much smaller than K^{W}.

The special form of equation (10) in the cylindrical coordinates in the described physical situation is

$$\begin{aligned}\frac{\partial}{\partial x} W(r,x) = \frac{E(r,x)}{U} + \frac{1}{U}\int_0^{\pi}\int_0^{x}\int_0^{\infty} &\left\{K^{\mathrm{W}}(x-\xi)\right.\\ &\times\left.\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] - F(x-\xi)\right\}\\ &\times W^*(r,r_0,x-\xi)r_0\,\mathrm{d}r_0\,\mathrm{d}\xi\,\mathrm{d}\varphi,\end{aligned} \tag{21}$$

where

$$r^2 = x_2^2 + x_3^2, \quad r_0^2 = \xi_2^2 + \xi_3^2, \quad x = x_1, \quad \xi = \xi_1,$$

φ is the angular coordinate and, according to (6),

$$W(r,x) = \int_0^{\pi}\int_0^{x}\int_0^{\infty} W^*(r,r_0,x-\xi)r_0\,\mathrm{d}r_0\,\mathrm{d}\xi\,\mathrm{d}\varphi. \tag{22}$$

If $\Psi(r) = W(r,0) \equiv 0$, the solution of the set (21) and (22), according to (11)–(13), can be expressed in the form

$$W(r,x) = \frac{1}{U}\int_0^{\pi}\int_0^{x}\int_0^{\infty} E(r_0,\xi) \times G(r,r_0,x-\xi)r_0\,\mathrm{d}r_0\,\mathrm{d}\xi\,\mathrm{d}\varphi, \tag{23}$$

where G is the Green function for the equation

$$\frac{\partial}{\partial x} G = \frac{1}{U}\left\{K^{\mathrm{W}}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - F\right\} G \tag{24}$$

and

$$E = 2K^{\mathrm{C}}(x)\left\{\frac{\partial\langle C(r,x)\rangle}{\partial r}\right\}^2. \tag{25}$$

The Green function for equation (24), if the variance of concentration is reflected from the surface $x_3 = 0$, is

$$G(r,r_0,x) = \frac{1}{\pi\sigma_{\mathrm{W}}^2(x)}\frac{\theta(x)}{\theta(0)} \times \exp\left\{-\frac{r^2+r_0^2}{2\sigma_{\mathrm{W}}^2(x)}\right\} \times I_0\left\{\frac{rr_0}{\sigma_{\mathrm{W}}^2(x)}\right\}, \tag{26}$$

where σ_{W} as well as the function σ_{c} introduced below are determined in terms of K^{W} and K^{C} by the relations

$$K^{\mathrm{W}} = \frac{U}{2}\frac{\mathrm{d}\sigma_{\mathrm{W}}^2}{\mathrm{d}x}, \quad K^{\mathrm{C}} = \frac{U}{2}\frac{\mathrm{d}\sigma_{\mathrm{C}}^2}{\mathrm{d}x}, \quad \sigma_{\mathrm{W}}(0) = \sigma_{\mathrm{C}}(0) = 0, \tag{27}$$

and

$$\theta(x) = \exp\left\{-\frac{1}{U}\int F(x)\,\mathrm{d}x\right\}.$$

Let the contaminant be reflected from the surface $x_3 = 0$, and the mean concentration be given and have the Gaussian distribution in transversal sections of the plume,

$$\langle C(r,x)\rangle = \frac{q}{\pi U[\sigma_{\mathrm{C}}^2(x) + \sigma_0^2]} \times \exp\left\{-\frac{r^2}{2[\sigma_{\mathrm{C}}^2(x) + \sigma_0^2]}\right\}, \tag{28}$$

where σ_0 is the linear scale characterizing the finite size of the contaminant source. Then, according to (23) and (26), we have

$$W(r,x) = \left(\frac{q}{\pi U}\right)^2 \int_0^x \phi(r,x,\xi)\,\mathrm{d}\xi, \qquad (29)$$

where

$$\phi = \left\{\frac{2\sigma_W^2(x-\xi)}{[\sigma_C^2(\xi)+\sigma_0^2]^2\sigma_s^4} + \frac{r^2}{[\sigma_C^2(\xi)+\sigma_0^2]\sigma_s^6}\right\} \times \frac{\mathrm{d}\sigma_C^2(\xi)}{\mathrm{d}\xi}\,\frac{\theta(x-\xi)}{\theta(0)}, \qquad (30)$$

$$\sigma_s^2 = \sigma_C^2(\xi)+\sigma_0^2+2\sigma_W^2(x-\xi).$$

Now introduce the quantity P, which equals the variance, integrated over a transversal section of the plume,

$$P(x) = \pi \int_0^\infty W(r,x) r\,\mathrm{d}r = M_1 \int_0^x \frac{1}{[\zeta(\xi)+1]^2}\,\frac{\mathrm{d}\zeta(\xi)}{\mathrm{d}\xi}\,\frac{\theta(x-\xi)}{\theta(0)}\,\mathrm{d}\xi, \qquad (31)$$

where

$$M_1 = \frac{\pi}{2\sigma_0^2}\left(\frac{q}{\pi U}\right)^2, \quad \zeta(x) = \sigma_C^2(x)\sigma_0^{-2}.$$

For the plume of marked fluid particles there are no effects of molecular diffusion and the dissipation of the variance is absent, $\theta(x) \equiv \theta(0)$. So the integrated variance,

$$P(x) = P_1(x) = M_1\frac{\zeta}{\zeta+1}, \qquad (32)$$

equals zero if $\zeta = 0$ and approaches the value M_1 in the limit $\zeta \to \infty$ (Fig. 1, curve 1), when the squared mean concentration (28) integrated over a transversal section of the plume

$$P_2(x) = \pi \int_0^\infty \langle C(r,x)\rangle^2 r\,\mathrm{d}r = \frac{M_1}{\zeta+1}, \qquad (33)$$

equals M_1 if $\zeta = 0$ and approaches zero if ζ grows to the infinity (Fig. 1, curve 2) in such a way, that

$$P_1(x)+P_2(x) = M_1 = \text{const.} \qquad (34)$$

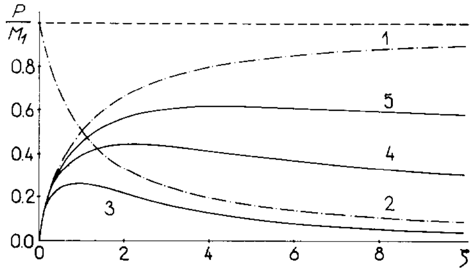

FIG. 1. The concentration variance (curves 1, 3–5) and the squared mean concentration (curve 2) integrated over a transversal section of the plume.

Results (32)–(34) are special cases of the general qualitative argument about the order of magnitude of the variance in turbulent fluid presented by Chatwin and Sullivan [2].

In all known experimental investigations into the turbulent diffusion of contaminant plumes carried out in natural water bodies, where molecular diffusion effects are present, the variance decrease downstream from the contaminant source has been observed. If the function $\theta(x)$ is given, for instance, in the form

$$\theta(x) = [\zeta(x)+1]^{-\alpha}, \qquad (35)$$

where $\alpha > 0$ is a given positive number and $\zeta(x)$ is a linear function, $\zeta(x-\xi) = \zeta(x)-\zeta(\xi)$, it will be easy to check up according to (31), that the dependence of the variance on the distance from the source can be divided into two monotonic phases. During the first phase the variance grows from zero at $x = 0$ (the contaminant injected into the turbulent environment is homogeneous) until it reaches some maximum value. In the second phase the variance decreases towards the asymptotical value of zero. The maximal variance and its location downstream from the source depends on the specific form of the function $\theta(x)$. So, for example, the higher peak of the variance situated farther downstream corresponds to the smaller α in (35), i.e. to the lower intensity of dissipation. [See curves 3 ($\alpha = 1.5$), 4 ($\alpha = 0.5$) and 5 ($\alpha = 0.2$) in Fig. 1].

In marine turbulence the first phase—the growth of the variance—most likely occurs only in the immediate vicinity of the contaminant source, in field experiments it has never been observed. In laboratory experiments conducted by Crum and Hanratty [8] the first phase was evidently present.

Suppose next that $\sigma_C^2 = \sigma_W^2$ are linear functions of x and $\theta(x) \equiv \theta(0)$. It follows from (29) that

$$W(r,x) = \left(\frac{q}{\pi U}\right)^2 \left\{\frac{\exp\left[\dfrac{-r^2}{2\sigma_C^2+\sigma_0^2}\right]}{\sigma_0^2(2\sigma_C^2+\sigma_0^2)} - \frac{\exp\left[\dfrac{-r^2}{\sigma_C^2+\sigma_0^2}\right]}{(\sigma_C^2+\sigma_0^2)^2}\right\}. \qquad (36)$$

The variance given by (36) can be interpreted as maximal [because of $\theta(x) \equiv \theta(0)$] in every space point if the conditions $\sigma_W^2 = \sigma_C^2 \sim x$ are valid. It is interesting to point out that the result (36) coincides with the expression for the variance got by Gifford [4] from a statistical model of a smoke plume, if σ_0^2 is regarded as the variance of the mean concentration distribution in transversal sections of the plume proper and σ_C^2 as the variance of the plume centreline meandering. Formula (36) gives variance values four times higher than Gifford's expression. This difference, the only one, is due to the variance and contaminant reflection from the surface $x_3 = 0$ in our case, whereas by Gifford's model the contaminant spreads in the whole infinite space without any reflection. Hence, from the position of the present model, Gifford's model describes the variance transformation in a steady axisymmetric contaminant plume in the absence of molecular diffusion effects, if the variance of spatial distributions of the mean concentration σ_C^2 and that of the concentration variance σ_W^2 are equal linear functions of x.

The dimensionless form of expression (36) is

$$\frac{W}{M_2}=\frac{\exp\left[-\dfrac{1+\zeta}{1+2\zeta}\rho^2\right]}{1+2\zeta}-\frac{\exp(-\rho^2)}{(1+\zeta)^2}, \tag{37}$$

where

$$M_2=q^2(\pi U\sigma_0^2)^{-2},\quad \rho^2=r^2(\sigma_C^2+\sigma_0^2)^{-1},\quad \zeta=\sigma_C^2\sigma_0^{-2}.$$

The dependence of WM_2^{-1} on ρ and ζ is shown in Fig. 2. It can be seen that initially a 'saddle-like' double-peaked distribution of the variance changes at $\zeta = 1.62$ into a 'bell-like' distribution. That kind of evolution of the variance profiles of a two-dimensional plume during the first phase of the variance transformation was observed by Crum and Hanratty [8] in a pipe flow with a Reynolds number 10^4.

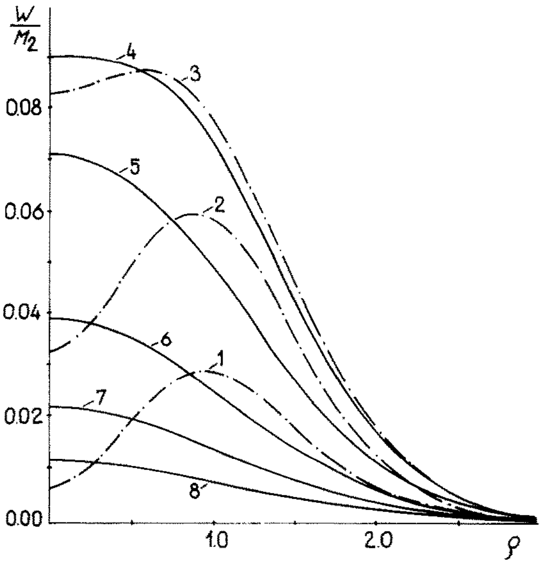

FIG. 2. The normed variance for $\zeta = 0.1$ (curve 1), 0.3 (curve 2), 1.0 (curve 3), 1.62 (curve 4), 4.0 (curve 5), 10 (curve 6), 20 (curve 7) and 40 (curve 8).

5. A STEADY PLUME AT REMOTE DISTANCES DOWNSTREAM

Field experiments carried out in the shallow Baltic with continuous plumes were described by Pavelson *et al.* [9]. The source of the contaminant was situated on the sea surface. The concentration of the contaminant (fluorescent dye) was measured at five transversal sections at five different downstream distances (25, 75, 150, 250 and 350 m) from the source. The distributions of the mean concentration were close to Gaussian distribution (28). The distribution of the intensity of concentration fluctuations $S = W^{1/2}\langle C\rangle^{-1}$ was obviously not self-similar. This is shown by points in Fig. 3 for $x = 25$ m (curve 1), 75 m (curve 2), 150 m (curve 3), 250 m (curve 4) and 350 m (curve 5). As we can see the distribution of S becomes steeper in the sections at farther distances downstream the contaminant source, when at the plume axis the intensity S decreases.

Function (29) was compared with the measured intensities by Toompuu [18]. The agreement appeared to be rather good (Fig. 3, curves 1–5) in this case the

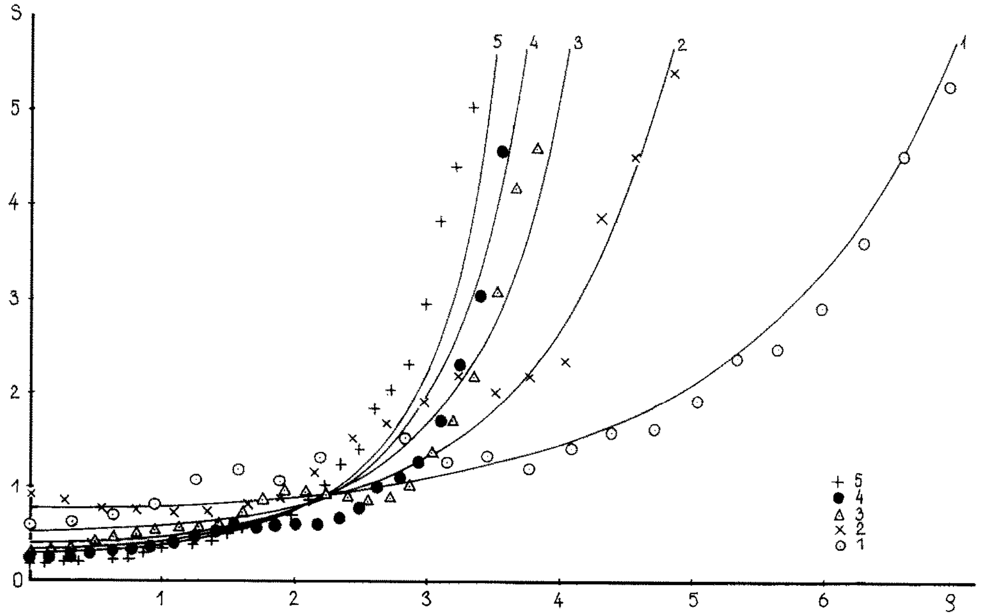

FIG. 3. The intensity of concentration fluctuations. The experimental points by Pavelson *et al.* [9], the curves are computed from (29).

phenomenological functions

$$F(x) = \frac{\gamma}{x+x_0}, \tag{38}$$

$$\sigma_W^2(x) = \sigma_C^2(x)\,[1-\tfrac{1}{2}\exp(-\beta x)], \tag{39}$$

where $x_0 = 5.6\times10^{-3}$ m, $\gamma = 1.76$ and $\beta = 3.8\times10^{-3}$ m^{-1} are constants, were applied, and if the following parameters of the mean concentration field were accepted

$$\sigma_C^2(x) = 2.5\times10^{-2}x^{1.3}\ \mathrm{m}^2, \tag{40}$$

$$\sigma_0 = 10^{-3}\ \mathrm{m}. \tag{41}$$

The first approximation (38) is almost completely the consequence of Csanady's [6] argument about the results of Gibson and Schwarz's [19] laboratory experiments on the decay of scalar field fluctuations in the grid turbulence, except for the parameter x_0 which was introduced to guarantee the boundness of the dissipation rate at $x = 0$. The approximation for σ_W^2 (39) was looked for on some heuristic considerations concerning the behaviour of σ_W^2 relative to σ_C^2 as $x \to \infty$ and $x \to 0$. It follows that the conditions

$$\sigma_W(\infty) = \sigma_C(\infty), \quad \sigma_W^2(0) = \tfrac{1}{2}\sigma_C^2(0), \tag{42}$$

which are fulfilled by expression (39), should be satisfied. In particular, from (42) it becomes obvious that K^W and K^C are not equal. Approximation (40) is simply the result of searching the dependence of the type $\sigma_C^2(x) = mx^n$ (m and n are the constants) in the set of experimental points by the least squares method. The last estimation (41) is for the effective final size of the contaminant source. It was derived from the equality condition of the total variance production rate in the 'real' contaminant source and in the 'Gaussian source', which is represented by function (28) if $x = 0$.

Now introduce the function

$$\psi(x,x'') = \pi\int_0^\infty W(r,x,x'')\,r\,\mathrm{d}r$$
$$= \pi\int_0^\infty\int_0^{x''}\phi(r,x,\xi)\,\mathrm{d}\xi\,r\,\mathrm{d}r, \tag{43}$$

where $\xi \leqslant x'' \leqslant x$, the normed values of which are shown in Fig. 4 for $x = 25$, 75 and 350 m and for $F(x)$ $\sigma_W^2(x)$, $\sigma_C^2(x)$, $\sigma_C^2(x)$ and σ_0 given by (38)–(41), curves 1–3.

From the figure it can be seen that for all the values of x under consideration more than 99.85% of the total variance is produced at the distances less than 0.1 m downstream from the contaminant source and that this contribution grows with x. Therefore, according to the present model the main part of the total variance is produced in the immediate vicinity of the contaminant source. The variance produced farther downstream from the source gives a negligibly small and even decrease with the increase of x contribution to the total amount of the variance at a given, sufficiently great distance from the source. The last conclusion is in full agreement with the result of the analysis of the order of magnitude of the terms of equation (1), made by Chatwin and Sullivan [3].

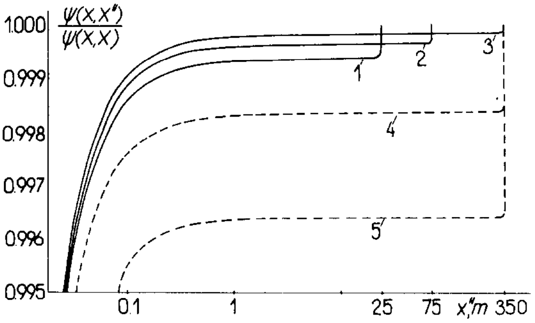

FIG. 4. Cumulative distribution of the variance for $\gamma = 1.76$ if $x = 25$ m (curve 1), 75 m (curve 2), 350 m (curve 3) and for $\gamma = 2.00$ (curve 4) and $\gamma = 2.08$ (curve 5) if $x = 350$ m.

The distribution of the contaminant in a contaminant source is determined by the nature of the source itself, whereas outside the boundaries of some initial plume region the special type of the contaminant distribution in the source no longer influences the concentration field sufficiently far downstream from the contaminant source. Sufficiently far from the contaminant source only some integral quantities of the source are essential. So, for instance, for the mean concentration field the essential integral quantity that does have an influence on the mean concentration field is the contaminant emission rate q. Our analysis shows that the concentration variance field is influenced not only by the contaminant emission rate q but also by the integral rate of the variance production near the source. This is the reason why the final size of the contaminant source on which the variance production rate near the source depends, should be taken into account when describing the variance field even at remote distances downstream from the source.

Using this argument, we simplify the problem and suppose that all the variance is produced in the point source, coinciding with the contaminant source in the origin of the coordinate axes. Express the production term in equation (24) as

$$E(r,x) = \frac{p}{\pi}\frac{\delta(r)}{r}\delta(x) \tag{44}$$

where p is the variance emission rate. The substitution of functions (44) and (26) into the general expression for solution (23) gives

$$W(r,x) = \frac{p}{\pi U\sigma_W^2(x)}\frac{\theta(x)}{\theta(0)}\exp\left\{-\frac{r^2}{2\sigma_W^2(x)}\right\}. \tag{45}$$

The intensity of the concentration fluctuations S, if $\sigma_C \gg \sigma_0$, can now be written as

$$S = \frac{W^{1/2}}{\langle C\rangle} = \kappa\Gamma(x)\exp\left\{\frac{\rho^2}{2}\left[1-\frac{\sigma_C^2(x)}{2\sigma_W^2(x)}\right]\right\}, \tag{46}$$

where

$$\rho = r\sigma_C^{-1}, \quad \kappa^2 = \frac{\pi U p}{q^2}, \quad \Gamma(x) = \left\{\frac{\theta(x)}{\theta(0)}\right\}^{1/2} \frac{\sigma_C^2(x)}{\sigma_W(x)}.$$

The values of S calculated from (46) together with (38)–(40), if $\kappa = 7.066 \times 10^2\ \mathrm{m}^{-1}$, practically coincide with the corresponding values of S calculated from (29) which were shown in Fig. 3.

Within the limits of the present model the profiles $S(\rho)$ as well as $W(\rho)$ can be considered self-similar far downstream from the source if the conditions

$$\sigma_W^2(x)\sigma_C^{-2}(x) = a, \quad \kappa\Gamma(x) = b \tag{47}$$

are satisfied. Here a and b are dimensionless constants independent of x. From (46) it follows that

$$S(\rho) = b \exp\left\{\frac{\rho^2}{2}\left(1 - \frac{1}{2a}\right)\right\}. \tag{48}$$

But, of course here the unanswered question arises: under what conditions do the functions $\sigma_W(x)$ and $\theta(x)$ satisfy the equalities (47). The above comparison of the model with the field experiment showed merely that the first equality in (47) may become satisfied sufficiently far downstream from the source and $a = 1$.

The only known experimental investigation the results of which formula (48) can be applied to, is the laboratory experiment carried out by Becker *et al.* [20]. The contaminant (oil fog) diffusion was measured in homogeneous turbulence in the central region of a pipe flow with the Reynolds number 6.84×10^5. The measured profiles of the intensity of concentration fluctuations are self-similar and shown in Fig. 5 for distances 20 (curve 1), 25 (curve 2), 35 (curve 3), 46 (curve 4) and 66 cm (curve 5) downstream from the source for the mean velocity 61 m s^{-1}. The agreement with S calculated from (48) for $a = b = 1$ is rather good. The decrease of the variance with the increasing Reynolds number observed in the experiment can be explained by the increase of the variance dissipation rate causing the parameter b to decrease in (48).

It should be noted that the diffusion time for the contaminant in the farthest downstream section of the plume in Becker *et al.*'s experiment was 0.011 s, whereas in the above-mentioned Crum and Hanratty experiment [8] it was almost sixfold—0.06 s. In spite of this the evolution of the variance in Crum and Hanratty's experiment took place only within the limits of the variance transformation first phase—the phase of the variance increase. Probably, on the higher turbulence intensity level (the Reynolds number in Becker *et al.*'s experiment exceeded that of Crum and Hanratty's by almost two orders of magnitude) the evolution of the variance passes the initial phases faster and reaches the phase of self-similarity earlier.

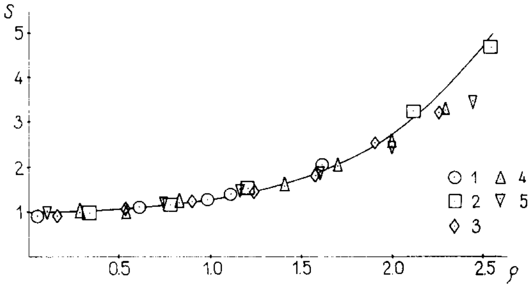

Fig. 5. The intensity of concentration fluctuations. The experimental points by Becker *et al.* [20], the curve is computed from (48) for $a = b = 1$.

Finally we point out that the distance downstream from which (47) and hence (48) become valid, should depend considerably on the variance dissipation rate. To illustrate the point the function $\psi(x, x'')$ for $x = 350$ m and $\gamma = 2.00$ (curve 4) and $\gamma = 2.08$ (curve 5), and with the other parameters left unchanged, is plotted in Fig. 4. It can be seen that the contribution of the variance produced in the vicinity of $x = x''$ to the total amount of the variance at a fixed x increases with γ and may become essential for sufficiently big γ.

6. SUMMARY

Results of some earlier proposed variance models follow from the above-constructed integro-differential variance transformation model in special cases. The model takes into account the general restrictions following from the order of magnitude analysis of the contaminant conservation equation terms and is in good agreement with the available results of experimental investigations.

The experiments the model was compared to were carried out either in laboratory turbulence or in small-scale marine turbulence where the inhomogeneities of the velocity field could be considered negligibly small. On the other hand, the integro-differential variance transport equation as well as the definite model equation (10) do not impose any constraints on the velocity field which the diffusion process occurs in. Therefore, probably the proposed approach would become useful for the variance transformation description also in a more general velocity field.

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UN MODELE PHENOMENOLOGIQUE POUR LA TRANSFORMATION DE VARIANCE D'UN SCALAIRE PASSIF DANS UN FLUIDE TURBULENT

Résumé—L'équation exacte de convection pour la variance de concentration d'un contaminant passif, dans un fluide turbulent incompressible, écrites par Corrsin [1], est exprimée sous la forme intégro-différentielle. On propose l'équation modèle phénoménologique intégro-différentielle pour la transformation espace-temps de la variance et la forme générale de sa solution. A partir de quelques exemples simples particuliers, on montre que les solutions de l'équation modèle suivent quelques restrictions générales précédemment obtenues [2, 3]. En liaison avec le modèle quelques approches antérieures [1, 4] peuvent être réinterprétées. Le modèle est en bon accord avec les données disponibles dans la réalité et dans les expériences de laboratoire.

EIN PHÄNOMENOLOGISCHES MODELL FÜR DIE PASSIVE SKALARE VARIANZTRANSFORMATION IN EINEM TURBULENTEN FLUID

Zusammenfassung—Es wird die exakte Transportgleichung für die passive kontaminante Konzentrationsvarianz in inkompressiblen turbulenten Fluiden, abgeleitet von Corrsin [1], in der integralen Differentialform dargestellt. Es wird die phänomenologische integrale Differential-Modellgleichung für die Varianz der Raum–Zeit-Transformation und die allgemeine Form ihrer Lösung vorgeschlagen. Basierend auf ein paar wenigen Spezialfällen wird gezeigt, daß die Lösungen der Modellgleichungen einigen früher ermittelten allgemeinen Einschränkungen [2, 3] folgen. Aufbauend auf dem neuen Modell, können einige frühere Annahmen [1, 4] zum Problem neu interpretiert werden. Das Modell stimmt mit experimentellen Daten gut überein.

ФЕНОМЕНОЛОГИЧЕСКАЯ МОДЕЛЬ ТРАНСФОРМАЦИИ ДИСПЕРСИИ ПАССИВНОЙ СКАЛЯРНОЙ ПРИМЕСИ В ТУРБУЛЕНТНОЙ ЖИДКОСТИ

Аннотация—Выведенное Корсиным [1] уравнение баланса дисперсии концентрации пассивной примеси выражено в интегро-дифференциальной форме. Предложено феноменологическое интегро-дифференциальное модельное уравнение пространственно-временной трансформации дисперсии и общий вид его решения. На основе нескольких простых частных случаев показано, что решения модельного уравнения согласуются с некоторыми ранее выведенными общими ограничениями [2, 3]. Исходя из сконструированной модели, показано, что некоторые ранее предложенные подходы [1, 4] к рассматриваемой задаче могут быть переистолкованы. Модель хорошо согласуется с доступными данными полевых и лабораторных опытов.